Fungible Parameter Estimates in Latent Curve Models

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Current Topics in the Theory and Application of Latent Variable Models
The Ohio State University
September 9-10, 2010
Overview

Fitting a statistical model:

- Observed data.
- A model of interest, including parameters to be estimated.
- A criterion to be optimized in fitting the model to data (e.g., least squares). Call it $F$.

Model is fit to data.

Yields parameter estimates and sample value of fitting criterion, $\hat{F}$.

Interpret parameter estimates and model fit.
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Model is fit to data.
Yields parameter estimates and sample value of fitting criterion, $\hat{F}$. Interpret parameter estimates and model fit.

*We know: Any change in parameter estimates will cause $\hat{F}$ to get worse.*
Consider: Specify some value $\hat{F}^*$ that is just slightly worse than $\hat{F}$.

Difference between $\hat{F}^*$ and $\hat{F}$ is so small as to be not of practical importance.

Question: What parameter estimates would yield $\hat{F}^*$?

These alternative estimates would be very slightly suboptimal in terms of model fit.
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Can we find such alternative estimates?

Yes.
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Usually an infinite number. All of these are \textit{fungible}, exchangeable, yielding same degree of model fit.
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   Usually an infinite number. All of these are *fungible*, exchangeable, yielding same degree of model fit.

What do they look like?
   May be very different from optimal estimates and from each other.
Implications?

- Parameter estimates that may be very different from optimal estimates might produce essentially same level of model fit.

- Raises concerns about interpretation of values of optimal parameter estimates.
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Outline

- Recent work on this topic in regression analysis.
- Extension to other modeling frameworks.
- Specific context of latent curve models, with example.
- Implications, future directions.
Fungible regression coefficients

Waller (2008) on fungible weights in multiple linear regression.

- Given OLS regression weights, $\hat{\beta}$, and SMC, $R^2$.
- Choose $R^*2$ slightly smaller than $R^2$.
- Can compute infinite number of alternative weight vectors, $\hat{\beta}^*$, that yield slightly suboptimal $R^*2$.
- These are *fungible* weights, all yielding same $R^*2$.
- Some $\hat{\beta}^*$ may be very different from $\hat{\beta}$ and very different from each other.
- Implications for parameter sensitivity, interpretation of optimal regression weights.
Example: Data from Kuncel, Hezlett, & Ones (2001).

\( N = 82,659 \) graduate students.

Dependent variable: Social science GPA.

Independent variables: GRE verbal, math, analytic. (V, M, A)

Least squares weights: \( \hat{\beta} = (.24, .14, .10) \)

Squared multiple correlation: \( R^2 = .176 \)
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Obtaining fungible regression weights: Specify \( R^{*2} = .171 \)

Find coefficients \( \hat{\beta}^* \) that would yield this \( R^{*2} \).

There will be an infinite number of solutions for \( \hat{\beta}^* \).
Six fungible weight vectors (From Table 1 of Waller, 2008):

<table>
<thead>
<tr>
<th></th>
<th>$\hat{\beta}_1^*$ (Verbal)</th>
<th>$\hat{\beta}_2^*$ (Math)</th>
<th>$\hat{\beta}_3^*$ (Analytic)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>OLS</strong></td>
<td>.24</td>
<td>.14</td>
<td>.10</td>
</tr>
<tr>
<td>$\hat{\beta}_1^*$ : high</td>
<td>.33</td>
<td>.12</td>
<td>.00</td>
</tr>
<tr>
<td>$\hat{\beta}_2^*$ : high</td>
<td>.22</td>
<td>.23</td>
<td>.01</td>
</tr>
<tr>
<td>$\hat{\beta}_3^*$ : high</td>
<td>.16</td>
<td>.07</td>
<td>.23</td>
</tr>
<tr>
<td>$\hat{\beta}_1^*$ : low</td>
<td>.13</td>
<td>.14</td>
<td>.19</td>
</tr>
<tr>
<td>$\hat{\beta}_2^*$ : low</td>
<td>.24</td>
<td>.03</td>
<td>.17</td>
</tr>
<tr>
<td>$\hat{\beta}_3^*$ : low</td>
<td>.31</td>
<td>.19</td>
<td>-.04</td>
</tr>
</tbody>
</table>
Notes about these fungible vectors of regression coefficients:

- In terms of variance accounted for, they are all equally good, and all just slightly worse than the OLS weights. In practical terms, just as good. ($R^2 = 0.176 \text{ v. } R^*^2 = 0.171$)

- Some of these fungible weights are very different from the OLS weights, implying very different interpretation.

- Some of these fungible weight vectors are very different from each other.

- Relative magnitude of regression weights not a good indicator of importance of independent variables.

- This phenomenon is not a function of standard errors of regression weights, which are very small in this example (large $N$).
Generalizing to other statistical models

This is a kind of *sensitivity analysis*: Imposing small perturbations on data or model and examining effect on results.

Usual perturbations: case deletion, parameter estimates, parameterization.
Generalizing to other statistical models

This is a kind of sensitivity analysis: imposing small perturbations on data or model and examining effect on results.

Usual perturbations: case deletion, parameter estimates, parameterization.

Current context: Perturb optimal fit, examine effect on parameter estimates.

Consider any statistical model to be fit to data by optimizing a fitting criterion, $F$.

$F$ is a closed form function of data and model parameters.

Best parameter estimates yield optimized value of $F$, called $\hat{F}$.

Find parameter estimates that yield slightly suboptimal value, $\hat{F}^*$. 
Extension to Structural Equation Modeling (SEM)

Recent work by MacCallum, Lee, and Browne:

• Define fungible parameter estimates in SEM.

• Develop and implement computational method for obtaining fungible estimates.

  Adapted computational method from previous work on isopower in SEM (MacCallum, Lee, & Browne, 2009).

• Application to various models: SEM, ARMA, Mediation.
Investigation in Latent Curve Models

To be presented:

• Brief overview of linear LCM.
• Empirical illustration.
• Define fungible parameter estimates in LCM.
• Computing these fungible estimates.
• Illustration of fungible estimates of selected parameters.
• Extension to conditional LCM.
Linear Latent Curve Model

A restricted common factor model with a mean structure.

Data: Scores for a sample of individuals on repeated measures of one outcome variable.

Model:
Two factors: intercept factor and slope factor (loadings fixed).
Parameters to be estimated: Factor means, variances, covariance; error variance at each occasion. (This is “random intercept, random slope” model.)
Empirical Illustration

Data from NICHD Study of Early Child Care; Used in book on LCM by Preacher, Wichman, MacCallum, & Briggs, 2008).

• Outcome variable: CLSN: A measure of closeness between mother and child (mother’s report, measured using 15-item rating scale)

• Occasions of measurement: Grade 1, 3, 4, 5, 6.

• Sample: 851 cases.

Observe downward trend in means.
Results of fitting LCM to CLSN data (by ML)

Model Fit:  \( \chi^2_{14} = 75.900 \quad p < .001 \)

\[ \hat{F} = 0.0893 \]

RMSEA = 0.072

NNFI = .096

Parameter Estimates (MLE):

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>SE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept mean</td>
<td>38.00</td>
<td>0.08</td>
</tr>
<tr>
<td>Intercept variance</td>
<td>2.98</td>
<td>0.29</td>
</tr>
<tr>
<td>Slope mean</td>
<td>-0.36</td>
<td>0.02</td>
</tr>
<tr>
<td>Slope variance</td>
<td>0.14</td>
<td>0.02</td>
</tr>
<tr>
<td>Intercept-Slope covariance</td>
<td>0.25</td>
<td>0.06</td>
</tr>
<tr>
<td>Error variance</td>
<td>3.70</td>
<td>0.10</td>
</tr>
</tbody>
</table>
Exploring fungible estimates in LCM for CLSN example

We know: If we perturb any of these parameter estimates, model fit will deteriorate (value of \( \hat{F} = 0.0893 \) will increase).

- Choose a value of \( \hat{F} \) just slightly larger than optimized value.
- Use RMSEA to guide this choice.
- Observed RMSEA = .072.
- Set perturbed RMSEA* = .077. Implies perturbed \( \hat{F}^* = 0.0997 \).
- Find parameter estimates that would produce \( \hat{F}^* = 0.0997 \).
LCM in example has 6 free parameters.

To simplify the problem, choose two parameters of interest. For example, mean and variance of slope factor.

- Hold all other parameter estimates at MLE.

- Find perturbed values of selected parameters that would produce perturbed model fit of $\hat{F}^* = 0.0997$. 
Slope Variance

ML solution for two parameters.

\( \hat{\theta} \)
Choose unit length direction vector, $d_n$. 

$\hat{\theta}$
Solve for scaling constant, $\kappa_n$. 

Slope Variance

Slope Mean

$\hat{\theta}$

$\kappa_n$
Define point representing fungible parameter estimates, $\hat{\theta}_n^* = \hat{\theta} + \kappa_n d_n$.
Repeat for direction vectors selected to cover parameter space.
Generate contour showing fungible estimates.
Fungible Estimates of Slope Parameters

MLE: (-.36, .37)
Fungible Estimates of Intercept Parameters

MLE: (38.0, 1.72)
Fungible Estimates of Intercept Parameters

Intercept Mean

Intercept Standard Deviation

MLE: (38.0, 1.72)

(38.2, 1.60)

(37.8, 1.87)
Extension to Conditional LCM

Introduce child’s gender as predictor of intercepts and slopes.

Introduces an additional variable and two additional parameters:

Coefficients for:

- Linear effect of gender on intercept.
- Linear effect of gender on slope.
Results of fitting conditional LCM:

Model fit: $\chi^2_{17} = 81.647 \quad p < .001$

$\hat{F} = 0.0961$

RMSEA = 0.067

NNFI = .096

Parameter Estimates (MLE):

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate (SE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept mean</td>
<td>38.14 (0.11)</td>
</tr>
<tr>
<td>Intercept variance</td>
<td>2.96 (0.28)</td>
</tr>
<tr>
<td>Slope mean</td>
<td>-0.33 (0.03)</td>
</tr>
<tr>
<td>Slope variance</td>
<td>0.14 (0.02)</td>
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<td>Intercept-Slope covariance</td>
<td>0.25 (0.06)</td>
</tr>
<tr>
<td>Error variance</td>
<td>3.70 (0.10)</td>
</tr>
<tr>
<td>Gender effect on slope</td>
<td>-0.05 (0.04)</td>
</tr>
<tr>
<td>Gender effect on intercept</td>
<td>-0.29 (0.16)</td>
</tr>
</tbody>
</table>
Obtain fungible estimates for effect of gender on slope, intercept.

- Hold all other parameter estimates at MLE.

- Observed model fit: \( \hat{F} = 0.0961 \quad \text{RMSEA} = 0.067 \)

- Set perturbed fit: Choose perturbed RMSEA* = .072.
  
  Implies perturbed \( \hat{F}^* = 0.1078 \)

- Find estimates of effects of gender that yield \( \hat{F}^* = 0.1078 \).
Fungible Estimates of Gender Effects

Gender Effect on Intercept

Gender Effect on Slope

MLE: (-.29, -.05)
Fungible Estimates of Gender Effects

Gender Effect on Intercept vs. Gender Effect on Slope

MLE: (-0.29, -0.05)

(-0.70, 0.05)

(.20, -0.15)
Summary and Implications

- We can define and compute fungible parameter estimates in LCM.

- Fungible estimates may be very different from optimal estimates and from each other.

  Depends on factors that include:
  - Flatness / steepness of discrepancy function surface.
  - Specified perturbation of optimal fit. (Should be small.)

- Interpretation of parameter estimates may be very different for different sets of fungible estimates.
• Results have implications for more or less rigorous interpretation of parameter estimates.

Small fungible contour implies more rigorous interpretation of parameter estimates is justified.

Large fungible contour implies risk in substantive interpretation.
Future Directions

• What to do with more than 2 or 3 parameters of interest.

• Specifying perturbation in model fit (Use $\chi^2$ ?).

• Factors that influence size of fungible contour.

• Remedies when fungible regions are large ????

• Generality of fungible-estimate phenomenon.

Phenomenon may be relevant across wide range of statistical models.

Can be examined in any modeling framework where a closed form fit function is being optimized.
Thank you.